The existence and stability conditions for periodic solutions need no modification.


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# ORBITAL STABILITY ANALYSIS USING FIRST INTEGRALS*

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A method is proposed for investigating the oribtal stability of periodic solutions of normal systems of ordinary differential equations. The Lyapunov function is derived from the first integrals of the equations of the perturbed motion and the scalar product of the velocity of motion along the orbit and the perturbation vector. Lypunov's second method was first used in connection with orbital stability in order to study the phase trajectories of systems with two degrees of freedom /1/.

1. ***Construction of the Lyapunov function.*** Let $\Omega \subset R^{n+1}$ be a domain containing the orbit /2/ of a $T$-periodic solution

$$Y = \Phi(t) \tag{1.1}$$

of the autonomous system

$$Y^{\cdot} = F(Y) \tag{1.2}$$

We shall investigate the orbital stability of (1.1) under the assumption that $F \in C^{(2)}(\Omega; R^{n+1})$.

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Let $\langle\cdot,\cdot\rangle$ denote the scalar product in $R^{n+1}$,

$$\psi(t)=\Phi^{\cdot}(t)\neq 0,\quad Z=\operatorname{col}(z_1,\ldots,z_{n+1})$$
$$\varkappa_1=\langle F(Z+\Phi(t)),\ \psi(t)\rangle^{-1},\quad \varkappa_2=\psi^2(t)-\langle\psi^{\cdot}(t),Z\rangle$$

It is well-known /2, Theorem 25/ that the solution (1.1) is orbitally stable if and only if the trivial solution of the equations

$$Z^{\cdot}=\varkappa_1\varkappa_2F(Z+\Phi(t))-\psi(t) \tag{1.3}$$

is stable in Lyapunov's sense to perturbations of $Z_0$ in the manifold $\langle\psi(0),Z_0\rangle=0$. Clearly, the following weaker assertion is also valid:

*Lemma.* If the trivial solution of system (1.3) is Lyapunov-stable, then the solution (1.1) of Eqs.(1.2) is orbitally stable.

Let us assume that Eqs.(1.2) have $m$ time-independent first integrals $U_i\in C^{(1)}(\Omega;R^1)$, such that

$$(\forall i:\ 1\leqslant i\leqslant m)(\forall Y\in\Omega):\langle\operatorname{grad}U_i(Y),F(Y)\rangle=0 \tag{1.4}$$

It follows from (1.4) that the first integrals

$$V_i=U_i(Z+\Phi(t))-U_i(\Phi(t)) \tag{1.5}$$

of the equations of the perturbed motion

$$Z^{\cdot}=F(Z+\Phi(t))-\psi(t) \tag{1.6}$$

are first integrals of system (1.3). Besides (1.5), Eqs.(1.3) also have the integral

$$V_{m+1}=\langle\psi(t),Z\rangle \tag{1.7}$$

(see /2/), for which there is no analogue in the context of system (1.2).

A trivial argument using Lyapunov's theorem derives the following result from the lemma.

*Theorem 1.* If one can construct from $V_1,\ldots,V_{m+1}$ a positive (or negative) definite function of $Z$, then solution (1.1) of system (1.2) is orbitally stable.

Thus, the only difference between the procedure for looking for a Lyapunov function in orbital stability analysis and the usual /3/ construction of a positive (negative) definite first integral of the equations of the perturbed motion (1.6) lies in the use of an additional function of the perturbations besides (1.5) - the linear form (1.7). This makes it possible to extend Chetayev's method of integral sheaves /4/ to orbital stability analysis.

**2. *Integral sheaves and orbital stability.*** Let the first integrals by $U_i\in C^{(3)}(\Omega;R^1)$. Put $l_i(t)=\operatorname{grad}U_i(\Phi(t))$ and express the functions (1.5) by means of Taylor's formula as

$$V_i=\langle l_i(t),Z\rangle+\langle Q_i(t)Z,Z\rangle+\varepsilon_i(t,Z)$$

The maps

$$l_i\in C^{(2)}(R^1;R^{n+1}),\quad Q_i\in C^{(1)}(R^1;L(R^{n+1},R^{n+1}))$$

are $T$-periodic, and $\varepsilon_i(t,Z)=O(\|Z\|^3)$ uniformly in $t\in R^1$.

*Theorem 2.* If there exist $\lambda_i\in R^1$ such that

$$\Sigma\lambda_il_i(t_0)=0 \tag{2.1}$$

at some time $t_0$, and the quadratic form

$$Q=\Sigma\lambda_i\langle Q_i(t)Z,Z\rangle$$

is definite on the manifold $N=\{(t,Z)\mid\langle l_1(t),Z\rangle=\ldots=\langle l_m(t),Z\rangle=\langle\psi(t),Z\rangle=0\}$, then the solution (1.1) of system (1.2) is orbitally stable.

Here and below, summation will always be over $i$ from $i=1$ to $i=m$.

*Proof.* The functions $\langle l_i(t),Z\rangle$ are first integrals of the variational equations for the solution (1.1) (see /5/). Consequently, $l_i(t)$ are solutions of the linear system adjoint to the variational system /6/. By the existence and uniqueness theorem, Eq.(2.1) is equivalent to

$$\Sigma\lambda_il_i(t)\equiv 0 \tag{2.2}$$

Choose the signs of $\lambda_i$ in such a way that the form $Q$ becomes positive definite on $N$. Put

$$B=[0;T]\times\{Z\mid\|Z\|=1\},\quad H=\langle\psi(t),Z\rangle^2+\Sigma\langle l_i(t),Z\rangle^2$$

$\alpha_j$ are positive constants.

Let $P\supset N\cap B$ be an open set on which $Q\geqslant\alpha_1$. On the compact set $B\setminus P$, which is disjoint from $N$, $H\geqslant 2\alpha_2$ and $Q\geqslant-\alpha_3$. Consequently, if $\mu=\alpha_2\alpha_3^{-1}$, the function $V_{m+1}^2+\mu\Sigma\lambda_iV_i+\Sigma V_i^2$ is positive definite, since by (2.2) it differs from the form $\mu Q+H$, which is positive definite with respect to $Z$, by a quantity of the order of $\|Z\|^3$. By Theorem 1, the solution (1.1) is orbitally stable, as required.

*Remarks*. 1. Why does the vector $\psi(t_0)$ not have to appear in (2.1)? To explain this, consider the scalar products of both sides of the equation

$$\sum \lambda_i l_i(t_0) + \lambda_{m+1}\psi(t_0) = 0 \tag{2.3}$$

with $\langle\psi(t_0), \psi(t_0)\rangle^{-1}\psi(t_0)$. Since

$$\langle l_i(t_0), \psi(t_0)\rangle = \langle \operatorname{grad} U_i(\Phi(t_0)), F(\Phi(t_0))\rangle = 0$$

we conclude from (2.3) that $\lambda_{m+1} = 0$. Thus (2.3) is equivalent to (2.1).

2. The conditions of Theorem 2 are not only sufficient but also necessary for the existence of a function $\eta \in C^{(2)}(R^{m+1}; R^1)$ such that $\eta(V_1, \ldots, V_{m+1})$ and its Hessian matrix at $Z = 0$ are definite*. (*Bryum A.Z., Investigation of periodic solutions of the equations of mechanics by Lyapunov's methods. Candidate Dissertation, Donetsk, 1985.

3. The time $t$ plays the role of a parameter in the formulation of Theorem 2, so that the question of whether the form $Q$ is definite on $N$ may be settled by known methods /7, 8/.

To illustrate Theorem 2, we carry out an orbital stability analysis for some periodic solutions of the equations of motion of a rigid body about a fixed point.

## 3. *Pendulum-like motions in a central Newtonian force field.*

If the Clebsch integrability conditions /9/ are satisfied, the equations of motion

$$Ap^{\cdot} = (B - C)(qr - \varepsilon\gamma'\gamma'') \quad (ABC,\ pqr,\ \gamma'\gamma''\gamma) \tag{3.1}$$

$$\gamma^{\cdot} = r\gamma' - q\gamma'' \quad (\gamma\gamma'\gamma'',\ rpq) \tag{3.2}$$

have first integrals

$$U_1 = Ap^2 + Bq^2 + Cr^2 + \varepsilon(A\gamma^2 + B\gamma'^2 + C\gamma''^2)$$

$$U_2 = Ap\gamma + Bq\gamma' + Cr\gamma'', \quad U_3 = \gamma^2 + \gamma'^2 + \gamma''^2$$

$$U_4 = A^2p^2 + B^2q^2 + C^2r^2 - \varepsilon(BC\gamma^2 + AC\gamma'^2 + AB\gamma''^2)$$

System (3.1), (3.2) has a solution depending on the parameter $h$ /9/:

$$p_0 = 0, \quad q_0 = 0, \quad r_0 = \varphi^{\cdot}, \quad \varphi^{\cdot 2} = G(\varphi) \tag{3.3}$$

$$\gamma_0 = \sin\varphi, \quad \gamma_0' = \cos\varphi, \quad \gamma_0'' = 0$$

$$\alpha = AC^{-1}, \quad \beta = BC^{-1}, \quad G(\varphi) = h - \varepsilon(\alpha\sin^2\varphi + \beta\cos^2\varphi)$$

We may assume without loss of generality that $\alpha \geqslant \beta$. The periodicity conditions for (3.3) are

$$h \in\ ]\varepsilon\beta; +\infty[\ \setminus \{\varepsilon\alpha\} \tag{3.4}$$

In the notation of Sect.2,

$$l_1 = 2\operatorname{col}(0, 0, C\varphi^{\cdot}, \varepsilon A\sin\varphi, \varepsilon B\cos\varphi, 0)$$

$$l_2 = \operatorname{col}(A\sin\varphi, B\cos\varphi, 0, 0, 0, C\varphi^{\cdot})$$

$$l_3 = 2\operatorname{col}(0, 0, 0, \sin\varphi, \cos\varphi, 0)$$

$$l_4 = 2C\operatorname{col}(0, 0, C\varphi^{\cdot}, -\varepsilon B\sin\varphi, -\varepsilon A\cos\varphi, 0)$$

Apart from a multiplicative constant,

$$\lambda_1 = -C^{-1}, \lambda_2 = 0, \quad \lambda_3 = \varepsilon(\alpha + \beta), \quad \lambda_4 = C^{-2}$$

Therefore the quadratic form $Q$ is

$$(\alpha^2 - \alpha) z_1^2 + (\beta^2 - \beta) z_2^2 + \varepsilon(\alpha + \beta - \alpha\beta - 1) z_6^2 \tag{3.5}$$

and the equations of the manifold $N$ are

$$\alpha\sin\varphi z_1 + \beta\cos\varphi z_2 + \varphi^{\cdot} z_6 = 0, \quad \sin\varphi z_4 + \cos\varphi z_5 = 0 \tag{3.6}$$

$$\varphi^{\cdot} z_3 - \varepsilon\beta\sin\varphi z_4 - \varepsilon\alpha\cos\varphi z_5 = 0$$

$$\varepsilon(\beta - \alpha)\sin\varphi\cos\varphi z_3 + \varphi^{\cdot}\cos\varphi z_4 - \varphi^{\cdot}\sin\varphi z_5 = 0$$

Acccording to Theorem 2 we must check whether the form (3.5) is definite on the manifold (3.6). Change variables, putting

$$z_7 = -\beta^{-1} \cos \varphi z_1 + \alpha^{-1} \sin \varphi z_2$$

Then it follows from (3.6) that

$$z_1 = -(\alpha^{-1}\varphi^{\cdot} \sin \varphi z_6 + \beta \cos \varphi\, z_7)$$
$$z_2 = -\beta^{-1}\varphi^{\cdot} \cos \varphi z_6 + \alpha \sin \varphi z_7$$
$$z_3 = z_4 = z_5 = 0$$

Restricted to the manifold (3.6), the form (3.5) becomes a function of two independent variables - $z_6$ and $z_7$:

$$\{\varphi^{\cdot 2} [1 - (\alpha^{-1} \sin^2 \varphi + \beta^{-1} \cos^2 \varphi)] + \varepsilon (\alpha + \beta - \alpha\beta - 1)\} z_6^2 + 2 (\alpha - \beta) \varphi^{\cdot} \sin \varphi \cos \varphi z_6 z_7 + \alpha\beta (\alpha\beta - \alpha \sin^2 \varphi - \beta \cos^2 \varphi) z_7^2 \tag{3.7}$$

The necessary and sufficient condition

$$(\alpha - 1)(\beta - 1)(h - \varepsilon\alpha\beta) > 0 \tag{3.8}$$

for a quadratic form with coefficients periodic in $\varphi$ to be definite, obtained from Sylvester's criterion, must be considered together with (3.4). Solving the system of inequalities (3.4) and (3.8) and using Theorem 2, we obtain the following.

*Theorem 3.* If the constant $U_{10}$ of the energy integral on the solution (3.3) satisfies any of the conditions

$$U_{10} \in\, ]\, \varepsilon B; +\infty\, [\setminus \{\varepsilon A\}, \quad B \leqslant A < C$$
$$U_{10} \in\, ]\, \varepsilon ABC^{-1}; +\infty [, \quad C < B \leqslant A$$
$$U_{10} \in\, ]\, \varepsilon B; \varepsilon ABC^{-1}\, [, \quad B < C < A$$

then the solution is orbitally stable.

Putting $\varepsilon = 0$, we obtain a well-known condition for the stability of permanent rotations of an Euler gyroscope about the major and minor axes of the inertia ellipsoid (see /10, para. 392/).

**4. *Delone case.*** Under the Kovalevskaya conditions, the dynamic Euler equations in non-dimensional variables are

$$2p^{\cdot} = qr, \quad 2q^{\cdot} = -rp - \gamma'', \quad r^{\cdot} = \gamma' \tag{4.1}$$

System (4.1), (3.2) has first integrals

$$U_1 = 2 (p^2 + q^2) + r^2 - 2\gamma, \quad U_2 = 2 (p\gamma + q\gamma') + r\gamma''$$
$$U_3 = \gamma^2 + \gamma'^2 + \gamma''^2, \quad U_4 = (p^2 - q^2 + \gamma)^2 + (2pq + \gamma')^2$$

and a particular solution

$$p_0 = h \sin \varphi, \quad q_0 = \varphi^{\cdot}, \quad r_0 = 2h \cos \varphi \tag{4.2}$$
$$\gamma_0 = q_0^2 - p_0^2, \quad \gamma_0' = -2p_0q_0, \quad \gamma_0'' = \sin (\varphi + \alpha)$$
$$\varphi^{\cdot 2} = G (\varphi) = \cos (\varphi + \alpha) - h^2 \sin^2 \varphi$$

satisfying the Delone integrability conditions /11/. Here $h \geqslant 0$ and $\alpha$ are parameters.

We shall assume from now on that the solution (4.2) is not a constant.

We have

$$\{\varphi \mid G = dG/d\varphi = 0, \quad d^2G/d\varphi^2 \geqslant 0\} \neq \varnothing$$

if and only if

$$|\sin \alpha| = (3 \sqrt{3}h)^{-1}(2h^2 + h_1)^{1/2}(4h^2 - h_1) \tag{4.3}$$
$$h^4 \in [^3/_4; 1], \quad h_1 = (4h^4 - 3)^{1/2}$$

If $h$ and $\alpha$ do not satisfy conditions (4.3), the solution (4.2) will be periodic.

In the notation of Sect.2,

$$l_1 = 2\,\mathrm{col}\,(2p_0,\ 2q_0,\ r_0,\ -1,\ 0,\ 0)$$
$$l_2 = \mathrm{col}\,(2\gamma_0,\ 2\gamma_0',\ \gamma_0'',\ 2p_0,\ 2q_0,\ r_0)$$
$$l_3 = 2\,\mathrm{col}\,(0,\ 0,\ 0,\ \gamma_0,\ \gamma_0',\ \gamma_0''),$$
$$l_4 = \mathrm{col}\,(0,\ 0,\ 0,\ 0,\ 0,\ 0)$$
$$\Phi = \mathrm{col}\,(p_0,\ q_0,\ r_0,\ \gamma_0,\ \gamma_0',\ \gamma_0'') \tag{4.4}$$

Apart from a multiplicative constant,

$$\lambda_1 = \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = 1$$

The quadratic form $Q$ is

$$\langle l_5, Z\rangle^2 + \langle l_6, Z\rangle^2 \tag{4.5}$$
$$l_5 = \mathrm{col}\,(2p_0,\ -2q_0,\ 0,\ 1,\ 0,\ 0), \quad l_6 = \mathrm{col}\,(2q_0,\ 2p_0,\ 0,\ 0,\ 1,\ 0)$$

This form is clearly definite on the manifold

$$\langle l_1, Z\rangle = \langle l_2, Z\rangle = \langle l_3, Z\rangle = \langle l_4, Z\rangle = \langle \Phi^{\cdot}, Z\rangle = 0$$

if and only if the vectors $l_1, l_2, l_3, l_5, l_6$ and $\Phi^{\cdot}$ are linearly independent at any instant of time. Calculations show that $\langle l_5, \Phi^{\cdot}\rangle \equiv \langle l_6, \Phi^{\cdot}\rangle \equiv 0$. Therefore, in order to apply Theorem 2, we have to verify the condition $\mathrm{rank}\,\{l_1, l_2, l_3, l_5, l_6\} \equiv 5$. Using standard linear algebra and Theorem 2, we obtain the following orbital stability criterion.

*Theorem 4.* If $h > 0$ then any periodic solution of Eqs.(4.1), (3.2) satisfying the Delone integrability conditions, other than the rest point, is orbitally stable.

**5. *Pendulum motions of a Kovalevskaya gyroscope.*** Eqs.(4.1), (3.2) have a particular solution

$$p_0 = 0, \quad q_0 = \varphi^{\cdot}, \quad r_0 = 0, \quad \varphi^{\cdot 2} = h + \cos\varphi$$
$$\gamma_0 = \cos\varphi, \quad \gamma_0' = 0, \quad \gamma_0'' = \sin\varphi \tag{5.1}$$

describing the motion of a Kovalevskaya gyroscope about the major axis of the ellipsoid of inertia. If $h \neq \pm 1$, this solution is periodic.

In the notation of Sect.2,

$$l_1 = \mathrm{col}\,(0,\ 4q_0,\ 0,\ -2,\ 0,\ 0), \quad l_2 = \mathrm{col}\,(2\gamma_0,\ 0,\ \gamma_0'',\ 0,\ 2q_0,\ 0)$$
$$l_3 = 2\,\mathrm{col}\,(0,\ 0,\ 0,\ \gamma_0,\ 0,\ \gamma_0''), \quad l_4 = 2h\,\mathrm{col}\,(0,\ 2q_0,\ 0,\ -1,\ 0,\ 0)$$

Apart from a multiplicative constant,

$$\lambda_1 = -h, \quad \lambda_2 = \lambda_3 = 0, \quad \lambda_4 = 1$$

The quadratic form $Q$ is

$$4\gamma_0 z_1^2 + 4(\gamma_0 + h) z_2^2 - h z_3^2 + z_4^2 + z_5^2 + 4q_0 z_1 z_5 - 4q_0 z_2 z_4 \tag{5.2}$$

and the equations of the manifold $N$ are

$$2q_0 z_2 - z_4 = 0, \quad 2\cos\varphi z_1 + \sin\varphi z_3 + 2q_0 z_5 = 0, \quad \cos\varphi z_4 + \sin\varphi z_6 = 0$$
$$-{}^1/_2 \sin\varphi z_2 - q_0 \sin\varphi z_4 + q_0 \cos\varphi z_6 = 0 \tag{5.3}$$

Analysis shows that the form (5.2) is definite under conditions (5.3) only for $h \in ]-1, 0[$. Using Theorem 2, we can state the result as follows.

*Theorem 5.* If $h \in ]-1, 0[$, then the solution (5.1) of system (4.1), (3.2) is orbitally stable.

*Remark.* The solution (5.1) is stationary with respect to $p, r, p^2 - q^2 + \gamma$ and $2pq + \gamma'$, so Theorem 5 implies the result of /12/ concerning Lyapunov-stability with respect to these quantities.

The above sufficient condition for orbital stability has a simple physical meaning: if $h \in ]-1, 0[$ and the system is performing pendulum oscillations, then the angle between the vector of the force of gravity and the barycentric axis remains acute throughout the motion.

**6. *The Bobylev-Steklov case.*** We conclude with a non-trivial example in which the sufficient conditions of Theorem 2 are also necessary. Consider the periodic solution

$$p_0 = \mathrm{const}, \quad q_0 = 0, \quad r_0 = \varphi^{\cdot}$$
$$\varphi^{\cdot 2} = 2k\cos\varphi + p_0^{-2}(1 - k^2 - p_0^4) \tag{6.1}$$

$$\gamma_0 = k\cos\varphi - p_0^2,\quad \gamma_0' = -k\sin\varphi,\quad \gamma_0'' = -p_0 r_0$$

of Eqs.(4.1), (3.2) satisfying the Bobylev-Steklov integrability conditions /11/. Here $p_0 \neq 0$ and $k \geqslant 0$ are parameters. The domain of admissible values of $p_0$ and $k$ is

$$|k - p_0^2| < 1 \tag{6.2}$$

*Theorem 6.* The solution (6.1) of system (4.1), (3.2) is orbitally stable if and only if

$$0 \leqslant k < p_0^2,\quad k^2 + 3p_0^4 < 1 \tag{6.3}$$

or

$$p_0^2 < k < p_0^2 + 1,\quad 1 < k^2 + 3p_0^4 \tag{6.4}$$

*Proof. Sufficiency.* In the notation of Sect.2, the quantities $l_1, l_2, l_3$ are determined by formulae (4.4) with $q_0 = 0$, and

$$l_4 = 2\,\mathrm{col}\,(2p_0^3 + 2p_0\gamma_0,\ 2p_0\gamma_0',\ 0,\ p_0^2 + \gamma_0,\ 0,\ \gamma_0')$$

The quadratic form is as follows:

$$Q = 6p_0^2 z_1^2 + 6p_0^2 z_2^2 + z_4^2 + z_5^2 + 4p_0 z_1 z_4 + 4p_0 z_2 z_5 - (p_0 z_3 + z_6)^2 \tag{6.5}$$

and the equations of the manifold $N$ become

$$2p_0 z_1 + r_0 z_3 - z_4 = 0,\quad 2\gamma_0 z_1 + 2\gamma_0' z_2 + \gamma_0'' z_3 + 2p_0 z_4 + r_0 z_6 = 0 \tag{6.6}$$

If $k > 0$ we put $\lambda_1 = -p_0^2$, $\lambda_2 = -2p_0$, $\lambda_3 = -1$, $\lambda_4 = 1$.
Then $Q$ becomes

$$(4p_0^2 + 2\gamma_0) z_1^2 + 4\gamma_0' z_1 z_2 - 2\gamma_0 z_2^2 - (p_0 z_3 + z_6)^2 \tag{6.7}$$

and $N$ is defined by (6.6) with the additional equations

$$\gamma_0 z_4 + \gamma_0' z_5 + \gamma_0'' z_6 = 0,\quad \gamma_0' z_3 + \gamma_0' r_0 z_4 - (p_0^2 + \gamma_0) r_0 z_5 - p_0 \gamma_0'' z_6 = 0 \tag{6.8}$$

Eliminating the dependent variables from (6.6) and (6.8), one can check the forms (6.5) and (6.7) for definiteness using Sylvester's criterion. The conditions thus obtained on the parameters are precisely the disjunction of (6.3) and (6.4). Thus, by Theorem 2, conditions (6.3) and (6.4) are sufficient for (6.1) to be orbitally stable.

*Necessity.* Following /13/, we put

$$U_0 = p_0^{-2}(3p_0^4 - k^2 + 1),\quad 4K_1 = a^4 - 2U_0 a^2 + 4\sqrt{2}p_0^{-1}(p_0^4 - k^2 + 1)a + 4(k^2 - 1) \tag{6.9}$$

$$H_2 = {}^1/_2\,(U_0 - 3a_0^2),\quad H_3 = {}^1/_2\,(-a_0^4 + U_0 a_0^2 - 2)$$

where $a_0$ is a multiple root of the polynomial $K_1$. The relation

$$4K_1 = (a - \sqrt{2}p_0)^2\,[a^2 + 2\sqrt{2}p_0 a + 2p_0^{-2}\,(k^2 - 1)]$$

shows that $a_0^2 = 2p_0^2$. Substituting this into (6.9), we obtain

$$H_2 H_3 = {}^1/_2\, p_0^2\,(p_0^4 - k^2)(1 - 3p_0^4 - k^2)$$

It is known /13/ that if $H_2 H_3 \leqslant 0$ the equations of the Kovalevskaya gyroscope have a solution which tends asymptotically to (6.1) as $t \to -\infty$. By (6.2), a necessary and sufficient condition for $H_2 H_3 > 0$ to be true is precisely the disjunction of (6.3) and (6.4). Consequently, if neither of conditions (6.3) and (6.4) is satisfied, the solution (6.1) is orbitally unstable. This completes the proof.

Conditions (6.3) and (6.4) determine the region in the plane of the parameters $k$, $p_0^2$ characterized by necessary and sufficient conditions for orbital stability of periodic solutions of the equations governing the motion of a Kovalevskaya gyroscope, on the assumption that the Bobylev-Steklov integrability conditions are satisfied (see the figure).

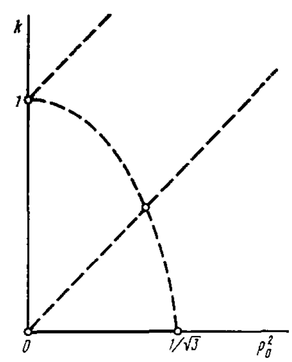


Fig.1

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# AN ALGORITHM FOR THE ASYMPTOTIC SOLUTION OF A SINGULARLY PERTURBED LINEAR TIME-OPTIMAL CONTROL PROBLEM*

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An algorithm for the approximate solution (in the asymptotic sense) of a singularly perturbed linear time-optimal control problem is proposed. A computational procedure is outlined, which permits the use of the resulting asymptotic approximation for the exact solution of the problem with a prescribed value of the small parameter.

*1. Statement of the problem.* In the class of scalar piecewise-continuous controls, we consider the following optimal control problem for a time-independent linear system:

$$x^{\cdot} = A(\mu) x + b(\mu) u, \quad x(0) = x^\circ, \quad x(T) = 0 \tag{1.1}$$
$$|u(t)| \leqslant 1, \quad J(u) = T \to \min$$
$$A(\mu) = \begin{Vmatrix} A_1/\mu & A_2/\mu \\ A_3 & A_4 \end{Vmatrix}, \quad b(\mu) = \begin{Vmatrix} b_1/\mu \\ b_2 \end{Vmatrix}, \quad x = \begin{Vmatrix} z \\ y \end{Vmatrix}, \quad x^\circ = \begin{Vmatrix} z^\circ \\ y^\circ \end{Vmatrix}$$

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